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Multilinear fractional integral with rough kernel on variable exponent Morrey-Herz spaces

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Abstract: In this article, we study a class of the multilinear fractional integral with rough kernel on Morrey-Herz space with $p(\cdot), q(\cdot), \alpha(\cdot)$. By using the properties of the variable exponent spaces, the boundedness of the multilinear fractional integral operator is obtained on variable nonhomogeneous Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$.

Keywords: Multilinear fractional integral, rough kernel, BMO function, Lipschitz function, Morrey-Herz spaces with variable exponents.

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1. Introduction

Suppose that $S^{n-1}(n>1)$ denote the unit sphere in $\mathbb{R}^n$ with the normalized Lebesgue measure $d\sigma(x')$. Let $\Omega \in L^s(S^{n-1})(1<s<\infty)$ be a homogeneous function of degree zero on $\mathbb{R}^n$. The multilinear fractional integral operator with rough kernel $T^A_{\Omega,\mu}(0<\mu<n)$ is defined by

$$T^A_{\Omega,\mu}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\mu+m}} R_{m+1}(A;x,y)f(y)dy,$$

where A is a function defined on $\mathbb{R}^n$ and $R_{m+1}(A;x,y)$ is the mth reminder of Taylor series of A at x about y. More precisely

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\gamma|\le m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma,$$

when $m=1$, $T^A_{\Omega,\mu}$ is just the commutators of the fractional integral with rough kernel with function A.

$$T^A_{\Omega,\mu}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\mu}} (A(x)-A(y))f(y)dy.$$

The multilinear fractional maximal operator with rough kernel is defined as

$$M^A_{\Omega,\mu}f(x) = \sup_{r>0} \frac{1}{r^{(n-\mu+m)}} \int\limits_{|x-y|<r} |\Omega(x-y)R_{m+1}(A;x,y)f(y)|dy.$$

In 1975, Coifman and Meyer [1] introduced multilinear integral and the boundedness of the multilinear fractional integral operator on Lebesgue spaces established in [2–4]. Li and Tao [5] discussed the boundedness of multilinear commutators with rough kernels on Morrey-Herz spaces.

On the other hand, the theory of the variable exponent function spaces has been rapidly developed after it was introduced by Kováčik and Rákosník [6]. After that, many researchers work in this direction has been done, see for example [7–13]. The boundedness of the multilinear fractional integral operator on variable Lebesgue spaces are established in [14,15]. Recently, Lu and Zhu [16] established the boundedness of the multilinear Calderón-Zygmund singular operators on Morrey-Herz spaces with variable exponents.

In this article, we study the boundedness of the multilinear fractional integral operator and multilinear fractional maximal operator with rough kernels on variable exponent Lebesgue spaces. The boundedness of the multilinear fractional integral operator is established on Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$.

Throughout this paper, let $|E|$ be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$ and χ_E be its characteristic function. We shall recall some definitions.

Definition 1. [7]. Let $p(\cdot) : E \to [1,\infty)$ be a measurable function, the variable exponent Lebesgue spaces $L^{p(\cdot)}(E)$ is defined as

$$L^{p(\cdot)}(E) = \{f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty \text{ for some constant } \eta > 0\}.$$

The space $L^{p(\cdot)}_{loc}(E)$ is defined as

$$L^{p(\cdot)}_{loc}(E) = \{\text{f is measurable} : f \in L^{p(\cdot)}(K) \; for\ all\ compact\ K \subset E\}.$$

The relation between Lebesgue spaces $L^{p(\cdot)}(E)$ and Banach spaces is defined as

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1\right\}.$$

We denote $p_- = \text{essinf}\,\{p(x) : x \in E\}$, $p_+ = \text{esssup}\{p(x) : x \in E\}$. Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Next, we give the definition of Morrey-Herz space with variable exponents $q(\cdot), p(\cdot), \alpha(\cdot)$. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{C_k}, k \in \mathbb{Z}$.

Definition 2. [18]. Let $q(\cdot), p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$. The nonhomogeneous Morrey-Herz space with variable exponents $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ is defined as

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda} \sum_{k=0}^{k_0} \left\|\left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1\right\}.$$

The homogeneous Morrey-Herz space with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ is defined as

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \{f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda} \sum_{k=-\infty}^{k_0} \left\|\left(\frac{2^{k\alpha(\cdot)}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \le 1\right\}.$$

2. Preliminaries

In this section, we give some properties of variable exponents that will be helpful in proving our main results.

Proposition 1. *[7]. If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies the follows inequalities:*

$$|p(x) - p(y)| \le \frac{-C}{log(|x-y|)}, |x-y| \le 1/2;$$
$$|p(x) - p(y)| \le \frac{C}{log(e+|x|)}, |y| \ge |x|.$$

then, we have $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$.

Proposition 2. *[14]. Suppose that* $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, $\Omega \in L^r(S^{n-1})$, $0 < \mu \le \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$, *then for all* $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$, *we have*

$$\|M_{\Omega,\mu} f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Now, we recall some lemmas.

Lemma 1. *[7].*

1. *Let* $p(\cdot) : \mathbb{R}^n \to [1, \infty)$, *for all function* f *and* g, *then*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

2. *If* $p(\cdot), q(\cdot), r(\cdot) \in \mathbb{R}^n$, $p(\cdot)$ *and* $\frac{1}{p(\cdot)} = \frac{1}{q(\cdot)} + \frac{1}{r(\cdot)}$. *Then there exists a constant* C *such that for all* $f \in L^{q(\cdot)}(\mathbb{R}^n), g \in L^{r(\cdot)}(\mathbb{R}^n)$, *we have*

$$\|fg\|_{L^{p(\cdot)}} \le C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{r(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2. *[7]. Let* $x \in \mathbb{R}^n$ *and* $\frac{1}{p(\cdot)} = \frac{1}{q} + \frac{1}{\tilde{q}(\cdot)}$, *then for all measurable function* f *and* g, *we have*

$$\|f(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|g(x)\|_{L^q(\mathbb{R}^n)} \|f(x)\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)}.$$

Lemma 3. *[9]. Suppose that* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ *and* $0 < p^- \le p^+ < \infty$, *then we have*

1. *for any cube* $|Q| \le 2^n$, *and all the* $\chi \in Q$, *we have* $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p(x)}$,
2. *for any cube* $|Q| \ge 1$, *we have* $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p_\infty}$, *where* $p_\infty = \lim_{x\to\infty} p(x)$.

Lemma 4. *[11]. If* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, *then there exist constants* $\delta_1, \delta_2, C > 0$ *such that for all balls* B *in* $\mathbb{R}^n$ *and all measurable subset* $S \subset B$, *we have*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Lemma 5. *[19]. For* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, *there exists constant* $C > 0$ *such that for any balls* B *in* $\mathbb{R}^n$, *we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 6. *[13]. Let* $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. *If* $f \in L^{p(\cdot)q(\cdot)}$, *then*

$$\min(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}) \le \||f|^{q(\cdot)}\|_{L^{p(\cdot)}} \le \max\left(\|f\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|f\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right).$$

Lemma 7. *[20]. Let* $b \in BMO(\mathbb{R}^n)$, *where* n *is a positive integer, and let the constant* $C > 0$. *Then for any* $l, j \in \mathbb{Z}$ *with* $l > j$, *we have*

1. $C^{-1}\|b\|_*^n \le \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^n \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*^n$,
2. $\|(b - b_{B_j})^n \chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(l-j)^n \|b\|_*^n \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Lemma 8. *[2]. For any* $\varepsilon > 0$ *with* $0 < \mu - \varepsilon < \mu + \varepsilon < n$, *we have*

$$|T^A_{\Omega,\mu} f(x)| \le C[M^A_{\Omega,\mu+\varepsilon} f(x)]^{\frac{1}{2}} [M^A_{\Omega,\mu-\varepsilon} f(x)]^{\frac{1}{2}}.$$

Lemma 9. *[4]. Let* A *be a function with derivatives of order* m *in* $\dot{\Lambda}_\beta (0 < \beta < 1)$. *Then there exists a constant* $C > 0$ *such that*

$$|R_{m+1}(A; x, y)| \le C\left(\sum_{|\gamma|=m} \|D^\gamma A\|_{\dot{\Lambda}_\beta}\right) |x - y|^{m+\beta}.$$

Lemma 10. *[21]. Let $b(x)$ be a function on $\mathbb{R}^n$ and $D^\gamma b \in L^q_{loc}(\mathbb{R}^n)$, where $q > n$, then*

$$|R_m(b,x,y)| \leq C|x-y|^m \sum_{|\gamma|=m} \left(\frac{1}{B(x,y)} \int_{B(x,y)} |D^\gamma b(z)| dz \right)^{\frac{1}{q}}.$$

where $B(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Lemma 11. *[7]. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q^+ < \infty$, then for any $s > 0$, we have*

$$\||f|^s\|_{q(\cdot)(\mathbb{R}^n)} = \|f\|^s_{sq(\cdot)(\mathbb{R}^n)}.$$

Lemma 12. *Let $D^\gamma A \in BMO(\mathbb{R}^n)(|\gamma| = |m|, m \geq 1)$, $\Omega \in L^r(S^{n-1})$, $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$, then we have*

$$\|M^A_{\Omega,\mu} f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$

where C is independent of f and A.

Proof. If $\widetilde{M}^A_{\Omega,\mu}$ is defined by:

$$\widetilde{M}^A_{\Omega,\mu} f(x) = \sup_{r>0} \frac{1}{r^{(n-\mu+m)}} \int_{\frac{r}{2}<|x-y|<r} |\Omega(x-y) R_{m+1}(A;x,y) f(y)| \mathrm{d}y.$$

Let $Q(x,r)$ be the cube centered at x and having diameter $5\sqrt{nr}$. If $\frac{r}{2} < |x-y| < r$, by Lemma 10, we have

$$\begin{aligned} |R_{m+1}(A;x,y)| = |R_{m+1}(A_k;x,y)| &\leq |R_m(A_k;x,y)| + \sum_{|\gamma|=m} \frac{1}{\gamma!} |D^\gamma A_k(y)||x-y|^m \\ &\leq C|x-y|^m \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} + \sum_{|\gamma|=m} |D^\gamma A(y) - m_{B_k}(D^\gamma A)| \right). \end{aligned}$$

By using Hölder's inequality, we get

$$\begin{aligned} \widetilde{M}^A_{\Omega,\mu} f(x) &= \sup_{r>0} \frac{1}{r^{n-\mu}} \int_{\frac{r}{2}<|x-y|<r} \left(\sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} + \sum_{|\gamma|=m} |D^\gamma A(y) - m_{B_k}(D^\gamma A)| \right) \\ &\quad \times |\Omega(x-y) f(y)| \mathrm{d}y \\ &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \left(M_{|\Omega|^t,\mu t}(|f|^t)(x) \right)^{\frac{1}{t}}. \end{aligned}$$

By the boundedness of the fractional maximal operator on $L^{p(\cdot)}(\mathbb{R}^n)$ spaces, we obtain that

$$\begin{aligned} \|\widetilde{M}^A_{\Omega,\mu} f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|M_{|\Omega|^t,\mu t}(|f|^t)(x)\|^{\frac{1}{t}}_{L^{\frac{p_2(\cdot)}{t}}} \\ &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \||f|^t\|^{\frac{1}{t}}_{L^{\frac{p_1(\cdot)}{t}}} \\ &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Since $M^A_{\Omega,\mu} f(x) \leq \widetilde{M}^A_{\Omega,\mu} f(x)$ for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \|M^A_{\Omega,\mu} f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq \|\widetilde{M}^A_{\Omega,\mu} f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Then, we get
$$\|M^A_{\Omega,\mu}f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$
This completes the proof of Lemma 12. □

Lemma 13. *Let $D^\gamma A \in BMO(\mathbb{R}^n)(|\gamma| = |m|, m \geq 1)$, $\Omega \in L^r(S^{n-1})$, $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ and $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$, then we have*
$$\|T^A_{\Omega,\mu}f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$
where C is independent of f and A.

Proof. Let $0 < \varepsilon < \min(\mu, n-\mu)$, and $r(\cdot) : \mathbb{R}^n \longrightarrow [1, +\infty)$, and let
$$\frac{1}{p_1(\cdot)} - \frac{1}{\frac{r(\cdot)p_2(\cdot)}{2}} = \frac{\mu-\varepsilon}{2},$$
$$\frac{1}{p_1(\cdot)} - \frac{1}{\frac{r'(\cdot)p_2(\cdot)}{2}} = \frac{\mu+\varepsilon}{2}.$$
By Lemma 8 and applying Hölder's inequality, we have
$$\|T^A_{\Omega,\mu}f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|(M^A_{\Omega,\mu+\varepsilon}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r'(\cdot)}}\|(M^A_{\Omega,\mu-\varepsilon}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r(\cdot)}}.$$
Since
$$\begin{aligned}\|(M^A_{\Omega,\mu-\varepsilon}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r(\cdot)}} &\leq C\|(M^A_{\Omega,\mu-\varepsilon}f)\|^{\frac{1}{2}}_{L^{\frac{p_2(\cdot)r(\cdot)}{2}}}\\ &\leq C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\right)^{\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$
Similar way, we concluded that
$$\|(M^A_{\Omega,\mu+\varepsilon}f)^{\frac{1}{2}}\|_{L^{p_2(\cdot)r'(\cdot)}} \leq C\left(\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\right)^{\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$
then, we have
$$\|T^A_{\Omega,\mu}f(x)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$
This completes the proof of Lemma 13. □

3. Main results

In this section, we investigate the boundedness of the multilinear fractional integral operator with rough kernel on variable nonhomogeneous Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$.

Theorem 1. *Suppose that $D^\gamma A \in BMO(\mathbb{R}^n)(|\gamma| = |m|, m \geq 1)$. Let $0 < \mu < n$, $\Omega \in L^r(s^{n-1})$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$, and $p_1(\cdot), p_2(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfy $0 < \mu \leq \frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$. If $(\lambda_1)(q_2)_+ = (\lambda_2)(q_1)_-$ and $\mu - n\delta_2 + (\lambda_1)/(q_1)_- < \alpha_+ < n\delta_1 + \frac{n}{r} + (\lambda_1)/(q_1)_-$. Then $T^A_{\Omega,\mu}$ is bounded from $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$ to $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $D^\gamma A \in BMO(\mathbb{R}^n)$, $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, we write
$$f(x) = \sum_{j=0}^{\infty} f(x)\chi_j(x) = \sum_{j=0}^{\infty} f_j(x).$$

By the definition of $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$, we have

$$\|T^A_{\Omega,\mu}(f)\chi_k\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T^A_{\Omega,\mu}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

For any $k_0\in z$, we see that

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T^A_{\Omega,\mu}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=0}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{11}+\eta_{12}+\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}},$$

where

$$\eta_{11}=\left\|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)}$$

$$=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\right\},$$

$$\eta_{12}=\left\|\sum_{k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)}$$

$$=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\right\},$$

$$\eta_{13}=\left\|\sum_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)}$$

$$=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

If $\eta=\eta_{11}+\eta_{12}+\eta_{13}$, thus

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq C.$$

That is

$$\|T^A_{\Omega,\mu}(f)\chi_k\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)} \leq C\eta \leq C[\eta_{11}+\eta_{12}+\eta_{13}].$$

Hence, it suffices to prove

$$\eta_{11},\eta_{12},\eta_{13} \leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)},$$

Denote $\eta_1 = \|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}$.

Now we consider η_{12} firstly. Applying Lemma 6, noting that $T^A_{\Omega,\mu}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, it follows

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right\|_{L^{p_2(\cdot)}}^{(q_2^1)_k}\\
&\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{(k-j)\alpha_+}2^{j\alpha_+}|T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right\|_{L^{p_2(\cdot)}}\right)^{(q_2^1)_k}\\
&\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^1)_k},
\end{aligned}$$

where

$$(q_2^1)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}> 1.\end{cases}$$

Since $f\in MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, we have

$$2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\leq 1.$$

From this and again applying Lemma 6, if $(q_1)_+\leq(q_2)_-$ and $\lambda_1(q_2)_+=\lambda_2(q_1)_-$, we can obtain that

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum\limits_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\left\|\frac{2^{k\alpha_+}|f\chi_k|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\right)^{(q_2^1)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1^1)_j}}\leq C\sum_{k=0}^{k_0}\left\{2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right\}^{\frac{(q_2^1)_k}{(q_1^1)_j}}\leq C,
\end{aligned}$$

where
$$(q_1^1)_j = \begin{cases} (q_1)_+, & \left\| \frac{2^{k\alpha_+}|f\chi_k|}{\eta_1} \right\|_{L^{p_1(\cdot)}} \leq 1, \\ (q_1)_-, & \left\| \frac{2^{k\alpha_+}|f\chi_k|}{\eta_1} \right\|_{L^{p_1(\cdot)}} > 1. \end{cases}$$
This implies that
$$\eta_{12} \leq C\eta_1 \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}.$$
Now we estimate of η_{11}. Let $x \in C_k$ $j \leq k-2$, then $|x-y| \sim |x|$, we can write that
$$|T^A_{\Omega,\mu} f_j(x)| \leq C \int_{C_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\mu+m}} |R_{m+1}(A;x,y)||f_j(y)|\mathrm{d}y,$$
where
$$A_k(x) = A(x) - \sum_{|\gamma|=m} \frac{1}{\gamma!}(D^\gamma A)x^\gamma,$$
and
$$R_{m+1}(A;x,y) = R_{m+1}(A_k;x,y),\ D^\gamma A_k(x) = D^\gamma A(x) - m_{\gamma_k}(D^\gamma A),\ |\gamma| = m.$$
Applying Lemma 10, we see that
$$\begin{aligned} |R_{m+1}(A;x,y)| &= |R_{m+1}(A_k;x,y)| \\ &\leq |R_m(A_k;x,y)| + \sum_{|\gamma|=m} \frac{1}{\gamma!}|D^\gamma A_k(y)||x-y|^m \\ &\leq C|x-y|^m \sum_{|\gamma|=m} \left(\left(\frac{1}{|\widetilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\gamma A_k(z)|^q \mathrm{d}z \right)^{\frac{1}{q}} + \frac{1}{\gamma!}|D^\gamma A_k(y)| \right) \\ &\leq C|x-y|^m \sum_{|\gamma|=m} \left(\|D^\gamma A\|_{BMO(\mathbb{R}^n)} + |D^\gamma A_k(y)| \right). \end{aligned}$$
Thus, we get
$$\begin{aligned} |T^A_{\Omega,\mu} f_j(x)| &\leq C \int_{C_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\mu}} \sum_{|\gamma|=m} \left[\|D^\gamma A\|_{BMO(\mathbb{R}^n)} + |D^\gamma A_k(y)| \right] |f_j(y)|\mathrm{d}y \\ &\leq C \sum_{|\gamma|=m} \int_{C_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\mu}} \left[\|D^\gamma A\|_{BMO(\mathbb{R}^n)} + |D^\gamma A_k(y)| \right] |f_j(y)|\mathrm{d}y \\ &\leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \int_{C_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\mu}} |f_j(y)|\mathrm{d}y \\ &\quad + \sum_{|\gamma|=m} \int_{C_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\mu}} |D^\gamma A_k(y)||f_j(y)|\mathrm{d}y \\ &=: L_1 + L_2. \end{aligned}$$

Applying the generalized Hölder's inequality, we have
$$L_1 \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f_j\|_{L^{p_1(\cdot)}} \left\| \frac{\Omega(x-y)}{|x-y|^{n-\mu}} \chi_j \right\|_{L^{p'_1(\cdot)}}.$$
If $\frac{1}{p_1(\cdot)} = \frac{1}{r} + \frac{1}{\widetilde{p'_1}(\cdot)}$, by Lemma 2, then we have
$$L_1 \leq \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f_j\|_{L^{p_1(\cdot)}} \|\Omega(x-y)\|_{L^r} \left\| \frac{\chi_j}{|x-y|^{n-\mu}} \right\|_{L^{\widetilde{p'_1}(\cdot)}}$$

$$\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}}\left[\int_{2^{k-2}}^{2^k} r^{n-1}\mathrm{d}r\int_{S^{n-1}}|\Omega(y')|^r\mathrm{d}\sigma(y')\right]^{\frac{1}{r}}$$

$$\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}}2^{\frac{kn}{r}}\|\Omega\|_{L^r(S^{n-1})}$$

$$\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}2^{-k(n-\mu)}2^{\frac{kn}{r}}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{\widetilde{p_1'}(\cdot)}}.$$

According to Lemma 3 and the formula $\frac{1}{\widetilde{p_1'}(\cdot)}=\frac{1}{p_1'(\cdot)}-\frac{1}{r}$, we have

$$\|\chi_{B_j}\|_{L^{\widetilde{p_1'}(\cdot)}}\approx\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}|B_j|^{\frac{-1}{r}}$$

Then, we obtain

$$L_1\leq\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

For L_2, applying the generalized Hölder's inequality, we get

$$L_2\leq C\sum_{|\gamma|=m}\|f_j\|_{L^{p_1(\cdot)}}\left\|\frac{\Omega(x-y)}{|x-y|^{n-\mu}}\chi_j|D^\gamma A_k(y)|\right\|_{L^{p'_1(\cdot)}}.$$

If $\frac{1}{p'_1(\cdot)}=\frac{1}{r}+\frac{1}{\widetilde{p_1'}(\cdot)}$, by Lemma 3, we have

$$\begin{aligned}L_2&\leq C\sum_{|\gamma|=m}2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\|\Omega(x-y)\|_{L^r}\left\|\chi_j|D^\gamma A_k(y)|\right\|_{L^{\widetilde{p_1'}(\cdot)}}\\&\leq C2^{-k(n-\mu)}\|f_j\|_{L^{p_1(\cdot)}}\|\Omega(x-y)\|_{L^r}\sum_{|\gamma|=m}\left\|\chi_j(D^\gamma A(x)-m_{\gamma_k}(D^\gamma A))\right\|_{L^{\widetilde{p_1'}(\cdot)}}.\end{aligned}$$

Similarly, and applying Lemma 7, we conclude that

$$L_2\leq\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}(k-j)2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

Combining the above two estimates about L_1, L_2, we obtain

$$|T^A_{\Omega,\mu}f_j(x)|\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}(k-j)2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

From this and using Lemma 6, we deduce that

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right\|_{L^{p_2(\cdot)}}^{(q_2^2)_k},$$

where

$$(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}\leq 1,\\ (q_2)_+, & \left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}>1.\end{cases}$$

Noting that if $\frac{1}{p_1(\cdot)}-\frac{1}{p_2(\cdot)}=\frac{\mu}{n}$, then $C_1|B|^{\frac{\mu}{n}}\|\chi_B\|_{L^{p_2(\cdot)}}\le\|\chi_B\|_{L^{p_1(\cdot)}}\le C_2|B|^{\frac{\mu}{n}}\|\chi_B\|_{L^{p_2(\cdot)}}$ (see[22], p.370). Therefore, together with this and applying Lemma 4 and Lemma 5, we have

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=0}^{k-2}2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}(k-j)\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=0}^{k-2}2^{-k(n-\mu)}2^{(k-j)\frac{n}{r}}(k-j)\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}|B_k|^{-\frac{\mu}{n}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=-\infty}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=0}^{k-2}(k-j)2^{(k-j)\frac{n}{r}}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[2^{k\alpha(\cdot)}\sum_{j=0}^{k-2}(k-j)2^{(j-k)(n\delta_1-\frac{n}{r})}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k-2}(k-j)2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)}\left\|\frac{2^{j\alpha_+}|f\chi_k|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)_k}.
\end{aligned}
$$

Since $f\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+=(\lambda_1)/(q_1)_-$ and $\alpha_+<n\delta_1+\frac{n}{r}+(\lambda_1)/(q_1)_-$, we have

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k-2}(k-j)2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)}\left\|\left(\frac{2^{j\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^2)_j}}_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right]^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k-2}(k-j)2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)}\left(2^{j\lambda}2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^2)_j}}\right]^{(q_2^2)_k}\\
&\le C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\\
&\times\left[\sum_{j=0}^{k-2}(k-j)2^{(j-k)((\lambda_1)/(q_1)_-+n\delta_1-\frac{n}{r}-\alpha_+)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^2)_j}}\right]^{(q_2^2)_k}\\
&\le C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=0}^{k-2}(k-j)2^{(j-k)((\lambda_1)/(q_1)_-+n\delta_1-\frac{n}{r}-\alpha_+)}\right]^{(q_2^2)_k}\le C,
\end{aligned}
$$

where

$$
(q_1^2)_j=\begin{cases}(q_1)_-, & \left\|\frac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\le 1,\\ (q_1)_+, & \left\|\frac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}>1.\end{cases}
$$

This implies that

$$\eta_{11} \leq C\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} = \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we estimate of η_{13}. Let $x \in C_k,\ j \geq k+2$, then $|x-y| \sim |y|$, by an argument similar to used in η_{11}, we have

$$|T^A_{\Omega,\mu} f_j(x)| \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)} (k-j) 2^{-j(n-\mu)} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

Applying Lemma 6, we have

$$2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{k+2}^{\infty} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \frac{2^{k\alpha(\cdot)} |\sum_{j=k+2}^{\infty} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)}} \right\|_{L^{p_2(\cdot)}}^{(q_2^3)_k},$$

where

$$(q_2^3)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k+2}^{\infty} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{p_2(\cdot)}} \leq 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k+2}^{\infty} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{p_2(\cdot)}} > 1. \end{cases}$$

Therefore, applying Lemma 4 and Lemma 5, we get

$$\begin{aligned}
&2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} 2^{-j(n-\mu)} (k-j) \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_2(\cdot)}} \right]^{(q_2^3)_k} \\
&\leq C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} 2^{-j(n-\mu)} (k-j) \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} 2^{-\mu k} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \right]^{(q_2^3)_k} \\
&\leq C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} 2^{(j-k)\mu} (k-j) \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} 2^{-jn} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \right]^{(q_2^3)_k} \\
&\leq C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} 2^{(j-k)\mu} (k-j) \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}} \right]^{(q_2^3)_k} \\
&\leq C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=k+2}^{\infty} (k-j) 2^{(k-j)(n\delta_2-\mu)} \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right]^{(q_2^3)_k}.
\end{aligned}$$

Notice that $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+ = (\lambda_1)/(q_1)_-$ and $\alpha_+ > \mu - n\delta_2 + (\lambda_1)/(q_1)_-$, we have

$$\begin{aligned}
&2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k+2}^{\infty} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{BMO(\mathbb{R}^n)}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\leq C 2^{-k_0\lambda_2} \sum_{k=0}^{\infty} \left[\sum_{j=k+2}^{\infty} (k-j) 2^{(k-j)(n\delta_2-\mu+\alpha_+)} \left\| \left(\frac{2^{j\alpha_+} |f\chi_k|}{\eta_1} \right)^{q_1(\cdot)} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}^{\frac{1}{(q_1^3)_j}} \right]^{(q_2^3)_k}
\end{aligned}$$

$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=k+2}^{\infty}(k-j)2^{(k-j)(n\delta_2-\mu+\alpha_+)}\left(2^{j\lambda}2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^3)_j}}\right]^{(q_2^3)_k}$$

$$\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}$$

$$\times\left[\sum_{j=k+2}^{\infty}(k-j)2^{(k-j)(n\delta_2-\mu-(\lambda_1)/(q_1)_-+\alpha_+)}\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^3)_j}}\right]^{(q_2^3)_k}$$

$$\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=k+2}^{\infty}(k-j)2^{(k-j)(n\delta_2-\mu-(\lambda_1)/(q_1)_--\alpha_+)}\right]^{(q_2^3)_k}\leq C,$$

where

$$(q_1^3)_j=\begin{cases}(q_1)_-, & \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\leq 1,\\ (q_1)_+, & \left\|\frac{2^{j\alpha_+}|f_{\chi_j}|}{\eta_1}\right\|_{L^{p_1(\cdot)}}>1.\end{cases}$$

This implies that

$$\eta_{13}\leq C\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}=\sum_{|\gamma|=m}\|D^\gamma A\|_{BMO(\mathbb{R}^n)}\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}.$$

The proof of Theorem 1 is finished. □

Theorem 2. *Suppose that $D^\gamma A\in\dot\wedge_\beta(|\gamma|=|m|,m\geq 1)$. Let $0<\mu<n,\Omega\in L^r(s^{n-1})$, $q_1(\cdot)$, $q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\geq(q_1)_+$, and $p_1(\cdot),p_2(\cdot)\in\mathfrak{B}(\mathbb{R}^n)$ satisfy $0<\mu+\beta\leq\frac{n}{(p_1)_+}$, $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\mu+\beta}{n}$. If $(\lambda_1)(q_2)_+=(\lambda_2)(q_1)_-$ and $(\mu+\beta)-n\delta_2+(\lambda_1)/(q_1)_-<\alpha_+<n\delta_1+\frac{n}{r}+(\lambda_1)/(q_1)_-$. Then $T^A_{\Omega,\mu}$ is bounded from $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$ to $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $D^\gamma A\in\dot\wedge_\beta(\mathbb{R}^n)$, $f\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, we write

$$f(x)=\sum_{j=0}^{\infty}f(x)\chi_j=\sum_{j=0}^{\infty}f_j(x).$$

By the definition of $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$, we have

$$\left\|T^A_{\Omega,\mu}(f)\chi_k\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p_2(\cdot)}(\mathbb{R}^n)}=inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T^A_{\Omega,\mu}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\right\}.$$

For any $k_0\in z$, we see that

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T^A_{\Omega,\mu}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{21}+\eta_{22}+\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}},$$

where

$$\eta_{21} = \left\| \sum_{j=0}^{k-2} T_{\Omega,\mu}^A(f_j)\chi_k \right\|_{MK_{q_2(\cdot),p_2(\cdot)}^{\alpha(\cdot),\lambda_2}(\mathbb{R}^n)}$$

$$= \inf\left\{ \eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T_{\Omega,\mu}^A(f_j)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq 1 \right\},$$

$$\eta_{22} = \left\| \sum_{k-1}^{k+1} T_{\Omega,\mu}^A(f_j)\chi_k \right\|_{MK_{q_2(\cdot),p_2(\cdot)}^{\alpha(\cdot),\lambda_2}(\mathbb{R}^n)}$$

$$= \inf\left\{ \eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k-1}^{k+1} T_{\Omega,\mu}^A(f_j)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq 1 \right\},$$

$$\eta_{23} = \left\| \sum_{j=k+2}^{\infty} T_{\Omega,\mu}^A(f_j)\chi_k \right\|_{MK_{q_2(\cdot),p_2(\cdot)}^{\alpha(\cdot),\lambda_2}(\mathbb{R}^n)}$$

$$= \inf\left\{ \eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k+2}^{\infty} T_{\Omega,\mu}^A(f_j)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq 1 \right\}.$$

And $\eta = \eta_{21} + \eta_{22} + \eta_{23}$, thus

$$2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |T_{\Omega,\mu}^A(f_j)\chi_k|}{\eta} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq C.$$

So, we have

$$\|T_{\Omega,\mu}^A(f)\chi_k\|_{MK_{q_2(\cdot),p_2(\cdot)}^{\alpha(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\eta \leq C[\eta_{21} + \eta_{22} + \eta_{23}].$$

Hence

$$\eta_{21}, \eta_{22}, \eta_{23} \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{\wedge_\beta} \|f\|_{MK_{q_1(\cdot),p_1(\cdot)}^{\alpha(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Denote $\eta_1 = \|f\|_{MK_{q_1(\cdot),p_1(\cdot)}^{\alpha(\cdot),\lambda_1}(\mathbb{R}^n)}$.

Now we consider η_{22} firstly. Noting that $T_{\Omega,\mu}^A$ is bounded on $L^{p(\cdot)}$ (Theorem 5 in [14]), as argued about η_{12} in proof of Theorem 1, we can get

$$2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=k-1}^{k+1} T_{\Omega,\mu}^A(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\wedge_\beta}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \leq C.$$

This implies that

$$\eta_{22} \leq C\eta_1 \leq C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\wedge_\beta} \|f\|_{M\dot K_{q_1(\cdot),p_1(\cdot)}^{\alpha(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

Next, we consider η_{21}. Let $x \in C_k$, $j \le k-2$, then $|x-y| \sim |x|$, we get

$$|T^A_{\Omega,\mu} f_j(x)| \le \int_{C_j} \frac{|\Omega(x-y) f_j(y)|}{|x-y|^{n-\mu+m}} |R_{m+1}(A;x,y)| \mathrm{d}y.$$

By Lemma 9, and applying Hölder's inequality, we have

$$\begin{aligned}|T^A_{\Omega,\mu} f_j(x)| &\le C \sum_{|\gamma|=m} \|D^\gamma A\|_{\Lambda_\beta} \int_{C_j} \frac{|\Omega(x-y) f_j(y)|}{|x-y|^{n-(\mu+\beta)}} \mathrm{d}y \\ &\le C \sum_{|\gamma|=m} \|D^\gamma A\|_{\Lambda_\beta} \|f_j\|_{L^{p_1(\cdot)}} \left\| \frac{\Omega(x-y)}{|x-y|^{n-(\mu+\beta)}} \chi_j \right\|_{L^{p'1(\cdot)}}.\end{aligned}$$

In the same way as we estimated L_1 in Theorem 1, we obtain that

$$|T^A_{\Omega,\mu} f_j(x)| \le C \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta} 2^{-k(n-(\mu+\beta))} 2^{(k-j)\frac{n}{r}} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}}.$$

Applying Lemma 6, we get that

$$2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta}} \right\|_{L^{p_2(\cdot)}}^{(q_2^1)_k}.$$

Where

$$(q_2^1)_k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta}} \right)^{q_2(\cdot)} \right\|_{L^{p_2(\cdot)}} \le 1, \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta}} \right)^{q_2(\cdot)} \right\|_{L^{p_2(\cdot)}} > 1. \end{cases}$$

Since $\frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{\mu+\beta}{n}$, then $\|\chi_B\|_{L^{p_2(\cdot)}} \le C 2^{-k(\mu+\beta)} \|\chi_B\|_{L^{p_1(\cdot)}}$ (see[22], P. 370). Therefore, applying Lemma 4 and Lemma 5, we deduce that

$$\begin{aligned}&2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)} |\sum_{j=0}^{k-2} T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1 \sum_{|\gamma|=m} \|D^\gamma A\|_{\dot\Lambda_\beta}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\le C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=0}^{k-2} 2^{-k(n-(\mu+\beta))} 2^{(k-j)\frac{n}{r}} \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \|\chi_{B_k}\|_{L^{p_2(\cdot)}} \right]^{(q_2^1)_k} \\
&\le C 2^{-k_0\lambda_2} \sum_{k=0}^{k_0} \left[2^{k\alpha(\cdot)} \sum_{j=0}^{k-2} 2^{-k(n-(\mu+\beta))} 2^{(k-j)\frac{n}{r}} \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} 2^{-k(\mu+\beta)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \right]^{(q_2^1)_k} \\
&\le C 2^{-k_0\lambda_2} \sum_{k=-\infty}^{\infty} \left[2^{k\alpha(\cdot)} \sum_{j=0}^{k-2} 2^{(k-j)\frac{n}{r}} \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}} \right]^{(q_2^1)_k} \\
&\le C 2^{-k_0\lambda_2} \sum_{k=0}^{\infty} \left[2^{k\alpha(\cdot)} \sum_{j=0}^{k-2} 2^{(j-k)(n\delta_1-\frac{n}{r})} \left\| \frac{|f_j|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right]^{(q_2^1)_k} \\
&\le C 2^{-k_0\lambda_2} \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k-2} 2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)} \left\| \frac{2^{j\alpha_+}|f\chi_k|}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right]^{(q_2^1)_k}\end{aligned}$$

$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k-2}2^{(j-k)(n\delta_1-\frac{n}{r}-\alpha_+)}\left\|\left(\frac{2^{j\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}^{\frac{1}{(q_1^1)_j}}\right]^{(q_2^1)_k}.$$

Notice that $f \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+ = (\lambda_1)/(q_1)_-$ and $\alpha_+ < n\delta_1 + \frac{n}{r} + (\lambda_1)/(q_1)_-$. In the same way as we estimated η_{11} in Theorem 1, we obtain that

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=0}^{k-2}2^{(j-k)((\lambda_1)/(q_1)_-+n\delta_1-\frac{n}{r}-\alpha_+)}\times\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^1)_j}}\right]^{(q_2^1)_k}\\
&\leq C,
\end{aligned}$$

where

$$(q_1^1)_j=\begin{cases}(q_1)_-, & \left\|\frac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\leq 1,\\ (q_1)_+, & \left\|\frac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}> 1.\end{cases}$$

This implies that

$$\eta_{21}\leq C\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}=\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\|f\|_{M\dot K^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}.$$

Finally, we consider η_{23}. Let $x\in C_k$, $j\geq k+2$, then $|x-y|\sim|y|$, we have

$$|T^A_{\Omega,\mu}f_j(x)|\leq\int_{C_j}\frac{|\Omega(x-y)||f_j(y)|}{|x-y|^{n-\mu+m}}|R_{m+1}(A;x,y)|\mathrm{d}y.$$

Applying Lemma 10 and Hölder's inequality, then we have

$$\begin{aligned}
|T^A_{\Omega,\mu}f_j(x)| &\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}\int_{C_j}\frac{|\Omega(x-y)||f_j(y)|}{|x-y|^{n-(\mu+\beta)}}\mathrm{d}y\\
&\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}2^{-j(n-(\mu+\beta))}\|f_j\|_{L^{p_1(\cdot)}}\left\|\Omega(x-y)\chi_j\right\|_{L^{p'_1(\cdot)}}\\
&\leq C\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}2^{-j(n-(\mu+\beta))}\|f_j\|_{L^{p_1(\cdot)}}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}.
\end{aligned}$$

From this and applying Lemma 4 and Lemma 5, we conclude that

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\Lambda_\beta}}\right\|_{L^{p_2(\cdot)}}^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{2^{-j(n-(\mu+\beta))}}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{2^{-j(n-(\mu+\beta))}}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{2^{-j(n-(\mu+\beta))}}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'_1(\cdot)}}2^{-k(\mu+\beta)}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)_k}
\end{aligned}$$

$$
\begin{aligned}
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{(j-k)(\mu+\beta)}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}2^{-jn}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{(j-k)(\mu+\beta)}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\right]^{(q_2^2)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left[2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+\beta))}\left\|\frac{|f_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)_k},
\end{aligned}
$$

where

$$
(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\left(\dfrac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\wedge_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}\leq 1,\\ (q_2)_+, & \left\|\left(\dfrac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\wedge_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}>1.\end{cases}
$$

Notice that $f\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)$, $(\lambda_2)/(q_2)_+=(\lambda_1)/(q_1)_-$ and $\alpha_+>(\mu+\beta)-n\delta_2+(\lambda_1)/(q_1)_-$, by the same argument as that of η_{13}, we have

$$
\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T^A_{\Omega,\mu}(f_j)\chi_k|}{\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\wedge_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+\beta)+\alpha_+)}\left\|\left(\frac{2^{j\alpha_+}|f\chi_k|}{\eta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^2)_j}}_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right]^{(q_2^2)_k}\\
&\leq C\sum_{k=0}^{\infty}2^{(k-k_0)\lambda_2}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+\beta)-(\lambda_1)/(q_1)_-+\alpha_+)}\right.\\
&\quad\times\left.\left(2^{-j\lambda}\sum_{n=0}^{j}\left\|\left(\frac{2^{n\alpha_+}|f\chi_n|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^2)_j}}\right]^{(q_2^2)_k}\\
&\leq C,
\end{aligned}
$$

where

$$
(q_1^2)_j=\begin{cases}(q_1)_-, & \left\|\dfrac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}\leq 1,\\ (q_1)_+, & \left\|\dfrac{2^{j\alpha_+}|f\chi_j|}{\eta_1}\right\|_{L^{p_1(\cdot)}}>1.\end{cases}
$$

This implies that

$$
\eta_{23}\leq C\eta_1\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\wedge_\beta}=\sum_{|\gamma|=m}\|D^\gamma A\|_{\dot\wedge_\beta}\|f\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p_1(\cdot)}(\mathbb{R}^n)}.
$$

Then, the proof of Theorem 2 is finished. □

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