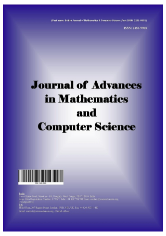

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Sum Formulas for Generalized Fifth-Order Linear Recurrence Sequences

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, closed forms of the summation formulas for generalized Pentanacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Pentanacci, Pentanacci-Lucas, fifth order Pell, fifth order Pell-Lucas, fifth order Jacobsthal and fifth order Jacobsthal-Lucas sequences. We present the proofs to indicate how these formulas, in general, were discovered. In fact, all the listed formulas of the special cases of of the main theorems may be proved by induction, but that method of proof gives no clue about their discovery.

Keywords: Pentanacci numbers; Pentanacci-Lucas numbers; sum formulas; summing formulas.

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1 Introduction

The generalized Pentanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2,\ W_3 = c_3, W_4 = c_4,\ n \geq 5 \end{aligned} \tag{1.1}$$

where W_0, W_1, W_2, W_3, W_4 are arbitrary real or complex numbers and r, s, t, u, v are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} + \frac{1}{v}W_{-n+5}$$

for $n = 1, 2, 3, ...$ when $v \neq 0$. Therefore, recurrence (1.1) holds for all integer n. Pentanacci sequence has been studied by many authors, see for example [1], [2], [3], [4].

For some specific values of W_0, W_1, W_2, W_3, W_4 and r, s, t, u, v it is worth presenting these special Pentanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s, t, u, v and initial values.

Table 1. A few members of generalized Pentanacci sequences

Sequences (Numbers)	Notation	OEIS [5]
Pentanacci	$\{P_n\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 1)\}$	A001591
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 1)\}$	A074048
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0, 1, 2, 5, 13; 2, 1, 1, 1, 1)\}$	A141448
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5, 2, 6, 17, 46; 2, 1, 1, 1, 1)\}$	-
fifth order Jacobsthal	$\{J_n^{(5)}\} = \{W_n(0, 1, 1, 1, 1; 1, 1, 1, 1, 2)\}$	A226310
fifth order Jacobsthal-Lucas	$\{j_n^{(5)}\} = \{W_n(2, 1, 5, 10, 20; 1, 1, 1, 1, 2)\}$	A226311

The first few values of the sequences with non-negative and negative indices are presented in the following table (Table 2).

Table 2. A few values of the sequences with positive subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	0	1	1	2	4	8	16	31	61	120	236	464	912	1793
Q_n	5	1	3	7	15	31	57	113	223	439	863	1695	3333	6553
$P_n^{(5)}$	0	1	2	5	13	34	89	232	605	1578	4116	10736	28003	73041
$Q_n^{(5)}$	5	2	6	17	46	122	315	821	2142	5588	14576	38018	99163	258650
$J_n^{(5)}$	0	1	1	1	1	4	9	17	33	65	132	265	529	1057
$j_n^{(5)}$	2	1	5	10	20	40	77	157	314	628	1256	2509	5021	10042

The first few values of the sequences with negative indices are presented in the following table (Table 3).

Table 3. A few values of the sequences with negative subscripts

n	1	2	3	4	5	6	7	8	9	10	11	12	13
P_{-n}	0	0	0	1	-1	0	0	0	2	-3	1	0	0
Q_{-n}	-1	-1	-1	-1	9	-7	-1	-1	-1	19	-23	5	-1
$P_{-n}^{(5)}$	0	0	0	1	-1	0	0	-1	4	-4	1	1	-7
$Q_{-n}^{(5)}$	-1	-1	-1	-5	14	-7	-1	3	-28	54	-34	1	38
$J_{-n}^{(5)}$	-1	0	$\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{8}$	$-\frac{17}{16}$	$-\frac{1}{32}$	$\frac{31}{64}$	$\frac{95}{128}$	$-\frac{33}{256}$	$-\frac{545}{512}$	$-\frac{33}{1024}$	$\frac{991}{2048}$
$j_{-n}^{(5)}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{11}{8}$	$\frac{13}{16}$	$\frac{13}{32}$	$\frac{13}{64}$	$\frac{13}{128}$	$-\frac{371}{256}$	$\frac{397}{512}$	$\frac{397}{1024}$	$\frac{397}{2048}$	$\frac{397}{4096}$

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(5)}$.

In this work, we investigate linear summation formulas of generalized Pentanacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [6, 7], see also [8]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [9,10], [11,12,13,14], [15,16], [17] and [18], respectively.

2 Linear Sum Formulas of Generalized Pentanacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with positive subscripts.

Theorem 2.1. *For $n \geq 0$ we have the following formulas:*

(a) *(Sum of the generalized Pentanacci numbers) If $r+s+t+u+v-1 \neq 0$ then*

$$\sum_{k=0}^{n} W_k = \frac{W_{n+5} + (1-r)W_{n+4} + (1-r-s)W_{n+3} + (1-r-s-t)W_{n+2} + (1-r-s-t-u)W_{n+1} + K_1}{r+s+t+u+v-1}$$

where

$$K_1 = -W_4 + (r-1)W_3 + (r+s-1)W_2 + (r+s+t-1)W_1 + (r+s+t+u-1)W_0.$$

(b) *If $(r-s+t-u+v+1)(r+s+t+u+v-1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\begin{array}{c}(1-s-u)W_{2n+2} + (t+v+rs+ru)W_{2n+1} + (t^2-u^2+v^2+rt+rv-su+2tv+u)W_{2n} \\ +(v+ru-sv+tu)W_{2n-1} + (v^2+rv+tv)W_{2n-2} + K_2\end{array}}{(r+s+t+u+v-1)(r-s+t-u+v+1)}$$

where

$$\begin{aligned} K_2 &= (s+u-1)W_4 - (t+v+rs+ru)W_3 + (r^2-s^2+rv-su+rt+2s+u-1)W_2 \\ &\quad +(sv-ru-tu-v)W_1 + (r^2-s^2+t^2-u^2+2rt+rv-2su+tv+2s+2u-1)W_0 \end{aligned}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\begin{array}{c}(r+t+v)W_{2n+2} + sW_{2n+1} + (-s^2+t^2+v^2-u^2+rv+rt-2su+2tv+u)W_{2n+1} \\ +(ru-st-sv+t+v)W_{2n} + (-u^2+v^2+rv-su+tv+u)W_{2n-1} \\ +(-sv-uv+v)W_{2n-2} + K_3\end{array}}{(r-s+t-u+v+1)(r+s+t+u+v-1)}$$

where

$$\begin{aligned} K_3 &= -(r+t+v)W_4 + (s+u+rt+rv+r^2-1)W_3 - (t+v+ru-st-sv)W_2 \\ &\quad +(2s+u+2rt+rv-su+tv+r^2-s^2+t^2-1)W_1 + (sv+uv-v)W_0. \end{aligned}$$

(c) *If* $r+t+v \neq 0 \wedge s+u-1=0$ *then*

$$\sum_{k=0}^{n} W_{2k} = \frac{W_{2n+1} + (t+v)W_{2n} + uW_{2n-1} + vW_{2n-2} - W_3 + rW_2 - uW_1 + (r+t)W_0}{r+t+v}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{W_{2n+2} + (t+v)W_{2n+1} + uW_{2n} + vW_{2n-1} - W_4 + rW_3 - uW_2 + (r+t)W_1}{r+t+v}.$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$vW_{n-5} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$\begin{aligned} vW_0 &= W_5 - rW_4 - sW_3 - tW_2 - uW_1 \\ vW_1 &= W_6 - rW_5 - sW_4 - tW_3 - uW_2 \\ vW_2 &= W_7 - rW_6 - sW_5 - tW_4 - uW_3 \\ vW_3 &= W_8 - rW_7 - sW_6 - tW_5 - uW_4 \\ &\vdots \\ vW_{n-4} &= W_{n+1} - rW_n - sW_{n-1} - tW_{n-2} - uW_{n-3} \\ vW_{n-3} &= W_{n+2} - rW_{n+1} - sW_n - tW_{n-1} - uW_{n-2} \\ vW_{n-2} &= W_{n+3} - rW_{n+2} - sW_{n+1} - tW_n - uW_{n-1} \\ vW_{n-1} &= W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1} - uW_n \\ vW_n &= W_{n+5} - rW_{n+4} - sW_{n+3} - tW_{n+2} - uW_{n+1} \end{aligned}$$

If we add the equations by side by, we obtain

$$\begin{aligned} v\sum_{k=0}^{n} W_k &= (W_{n+5} + W_{n+4} + W_{n+3} + W_{n+2} + W_{n+1} - W_4 - W_3 - W_2 - W_1 - W_0 + \sum_{k=0}^{n} W_k) \\ &\quad -r(W_{n+4} + W_{n+3} + W_{n+2} + W_{n+1} - W_3 - W_2 - W_1 - W_0 + \sum_{k=0}^{n} W_k) \\ &\quad -s(W_{n+3} + W_{n+2} + W_{n+1} - W_2 - W_1 - W_0 + \sum_{k=0}^{n} W_k) \\ &\quad -t(W_{n+2} + W_{n+1} - W_1 - W_0 + \sum_{k=0}^{n} W_k) - u(W_{n+1} - W_0 + \sum_{k=0}^{n} W_k) \end{aligned}$$

and then we get (a).

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we obtain

$$\begin{aligned}
rW_3 &= W_4 - sW_2 - tW_1 - uW_0 - vW_{-1} \\
rW_5 &= W_6 - sW_4 - tW_3 - uW_2 - vW_1 \\
rW_7 &= W_8 - sW_6 - tW_5 - uW_4 - vW_3 \\
rW_9 &= W_{10} - sW_8 - tW_7 - uW_6 - vW_5 \\
&\vdots \\
rW_{2n-1} &= W_{2n} - sW_{2n-2} - tW_{2n-3} - uW_{2n-4} - vW_{2n-5} \\
rW_{2n+1} &= W_{2n+2} - sW_{2n} - tW_{2n-1} - uW_{2n-2} - vW_{2n-3} \\
rW_{2n+3} &= W_{2n+4} - sW_{2n+2} - tW_{2n+1} - uW_{2n} - vW_{2n-1} \\
rW_{2n+5} &= W_{2n+6} - sW_{2n+4} - tW_{2n+3} - uW_{2n+2} - vW_{2n+1}.
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
r(-W_1 + \sum_{k=0}^{n} W_{2k+1}) &= (W_{2n+2} - W_2 - W_0 + \sum_{k=0}^{n} W_{2k}) \\
&\quad -s(-W_0 + \sum_{k=0}^{n} W_{2k}) - t(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1}) \\
&\quad -u(-W_{2n} + \sum_{k=0}^{n} W_{2k}) - v(-W_{2n+1} - W_{2n-1} + W_{-1} + \sum_{k=0}^{n} W_{2k+1}).
\end{aligned}$$

Since

$$W_{-1} = -\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4$$

we obtain

$$\begin{aligned}
r(-W_1 + \sum_{k=0}^{n} W_{2k+1}) &= (W_{2n+2} - W_2 - W_0 + \sum_{k=0}^{n} W_{2k}) \qquad (2.1) \\
&\quad -s(-W_0 + \sum_{k=0}^{n} W_{2k}) - t(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1}) - u(-W_{2n} + \sum_{k=0}^{n} W_{2k}) \\
&\quad -v(-W_{2n+1} - W_{2n-1} + (-\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4) + \sum_{k=0}^{n} W_{2k+1}).
\end{aligned}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5}$$

we write the following obvious equations;

$$\begin{array}{rcl} rW_2 &=& W_3 - sW_1 - tW_0 - uW_{-1} - vW_{-2} \\ rW_4 &=& W_5 - sW_3 - tW_2 - uW_1 - vW_0 \\ rW_6 &=& W_7 - sW_5 - tW_4 - uW_3 - vW_2 \\ rW_8 &=& W_9 - sW_7 - tW_6 - uW_5 - vW_4 \\ && \vdots \\ rW_{2n-2} &=& W_{2n-1} - sW_{2n-3} - tW_{2n-4} - uW_{2n-5} - vW_{2n-6} \\ rW_{2n} &=& W_{2n+1} - sW_{2n-1} - tW_{2n-2} - uW_{2n-3} - vW_{2n-4} \\ rW_{2n+2} &=& W_{2n+3} - sW_{2n+1} - tW_{2n} - uW_{2n-1} - vW_{2n-2} \\ rW_{2n+4} &=& W_{2n+5} - sW_{2n+3} - tW_{2n+2} - uW_{2n+1} - vW_{2n}. \end{array}$$

Now, if we add the above equations by side by, we obtain

$$\begin{array}{rcl} r(-W_0 + \sum\limits_{k=0}^{n} W_{2k}) &=& (-W_1 + \sum\limits_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1} + \sum\limits_{k=0}^{n} W_{2k+1}) \\ && -t(-W_{2n} + \sum\limits_{k=0}^{n} W_{2k}) - u(-W_{2n+1} - W_{2n-1} + W_{-1} + \sum\limits_{k=0}^{n} W_{2k+1}) \\ && -v(-W_{2n} - W_{2n-2} + W_{-2} + \sum\limits_{k=0}^{n} W_{2k}). \end{array}$$

Since

$$\begin{array}{rcl} W_{-1} &=& -\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4 \\ W_{-2} &=& -\frac{u}{v}(-\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4) - \frac{t}{v}W_0 - \frac{s}{v}W_1 - \frac{r}{v}W_2 + \frac{1}{v}W_3 \end{array}$$

we have

$$\begin{array}{rcl} r(-W_0 + \sum\limits_{k=0}^{n} W_{2k}) &=& (-W_1 + \sum\limits_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1} + \sum\limits_{k=0}^{n} W_{2k+1}) \qquad (2.2) \\ && -t(-W_{2n} + \sum\limits_{k=0}^{n} W_{2k}) - u(-W_{2n+1} - W_{2n-1} \\ && +(-\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4) + \sum\limits_{k=0}^{n} W_{2k+1}) - v(-W_{2n} - W_{2n-2} \qquad (2.3) \\ && +(-\frac{u}{v}(-\frac{u}{v}W_0 - \frac{t}{v}W_1 - \frac{s}{v}W_2 - \frac{r}{v}W_3 + \frac{1}{v}W_4) - \frac{t}{v}W_0 - \frac{s}{v}W_1 - \frac{r}{v}W_2 + \frac{1}{v}W_3) \\ && + \sum\limits_{k=0}^{n} W_{2k}). \end{array}$$

Then, solving the system (2.1)-(2.2), the required result of (b) and (c) follow.

Taking $r = s = t = u = v = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.1. *If* $r = s = t = u = v = 1$ *then for* $n \geq 0$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+5} - W_{n+3} - 2W_{n+2} - 3W_{n+1} - W_4 + W_2 + 2W_1 + 3W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{8}(-W_{2n+2} + 4W_{2n+1} + 5W_{2n} + 2W_{2n-1} + 3W_{2n-2} + W_4 - 4W_3 + 3W_2 - 2W_1 + 5W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{8}(3W_{2n+2} + 4W_{2n+1} + W_{2n} + 2W_{2n-1} - W_{2n-2} - 3W_4 + 4W_3 - W_2 + 6W_1 + W_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$).

Corollary 2.2. *For $n \geq 0$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+5} - P_{n+3} - 2P_{n+2} - 3P_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2} + 4P_{2n+1} + 5P_{2n} + 2P_{2n-1} + 3P_{2n-2} - 3)$.

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2} + 4P_{2n+1} + P_{2n} + 2P_{2n-1} - P_{2n-2} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 1, Q_2 = 3, Q_3 = 7, Q_4 = 15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 2.3. *For $n \geq 0$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+5} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} + 5)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 4Q_{2n+1} + 5Q_{2n} + 2Q_{2n-1} + 3Q_{2n-2} + 19)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + 4Q_{2n+1} + Q_{2n} + 2Q_{2n-1} - Q_{2n-2} - 9)$.

Taking $r = 2, s = t = u = v = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2. *If $r = 2, s = t = u = v = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{5}(W_{n+5} - W_{n+4} - 2W_{n+3} - 3W_{n+2} - 4W_{n+1} - W_4 + W_3 + 2W_2 + 3W_1 + 4W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{15}(-W_{2n+2}+6W_{2n+1}+7W_{2n}+3W_{2n-1}+4W_{2n-2}+W_4-6W_3+8W_2-3W_1+11W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{15}(4W_{2n+2}+6W_{2n+1}+2W_{2n}+3W_{2n-1}-W_{2n-2}-4W_4+9W_3-2W_2+12W_1+W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13$).

Corollary 2.4. *For $n \geq 0$, fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} P_k = \frac{1}{5}(P_{n+5} - P_{n+4} - 2P_{n+3} - 3P_{n+2} - 4P_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{15}(-P_{2n+2} + 6P_{2n+1} + 7P_{2n} + 3P_{2n-1} + 4P_{2n-2} - 4)$.

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{15}(4P_{2n+2} + 6P_{2n+1} + 2P_{2n} + 3P_{2n-1} - P_{2n-2} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 5, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{5}(Q_{n+5} - Q_{n+4} - 2Q_{n+3} - 3Q_{n+2} - 4Q_{n+1} + 9)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{15}(-Q_{2n+2} + 6Q_{2n+1} + 7Q_{2n} + 3Q_{2n-1} + 4Q_{2n-2} + 41)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{15}(4Q_{2n+2} + 6Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} - Q_{2n-2} - 14)$.

Taking $r = 1, s = 1, t = 1, u = 1, v = 2$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.3. *If $r = 1, s = 1, t = 1, u = 1, v = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = \frac{1}{5}(W_{n+5} - W_{n+3} - 2W_{n+2} - 3W_{n+1} - W_4 + W_2 + 2W_1 + 3W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{15}(-W_{2n+2}+5W_{2n+1}+11W_{2n}+2W_{2n-1}+8W_{2n-2}+W_4-5W_3+4W_2-2W_1+7W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{15}(4W_{2n+2}+10W_{2n+1}+W_{2n}+7W_{2n-1}-2W_{2n-2}-4W_4+5W_3-W_2+8W_1+2W_0)$.

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 2.6. *For $n \geq 0$, fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} J_k = \frac{1}{5}(J_{n+5} - J_{n+3} - 2J_{n+2} - 3J_{n+1} + 2)$.

(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{15}(-J_{2n+2} + 5J_{2n+1} + 11J_{2n} + 2J_{2n-1} + 8J_{2n-2} - 2)$.

(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{15}(4J_{2n+2} + 10J_{2n+1} + J_{2n} + 7J_{2n-1} - 2J_{2n-2} + 8)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20$).

Corollary 2.7. *For $n \geq 0$, fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} j_k = \frac{1}{5}(j_{n+5} - j_{n+3} - 2j_{n+2} - 3j_{n+1} - 7)$.

(b) $\sum_{k=0}^{n} j_{2k} = \frac{1}{15}(-j_{2n+2} + 5j_{2n+1} + 11j_{2n} + 2j_{2n-1} + 8j_{2n-2} + 2)$.

(c) $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{15}(4j_{2n+2} + 10j_{2n+1} + j_{2n} + 7j_{2n-1} - 2j_{2n-2} - 23)$.

3 Linear Sum Formulas of Generalized Pentanacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas of generalized Pentanacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$ we have the following formulas:*

(a) *(Sum of the generalized Pentanacci numbers with negative indices) If $r + s + t + u + v - 1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\begin{array}{c} -W_{-n+4} + (r-1)W_{-n+3} + (r+s-1)W_{-n+2} + (r+s+t-1)W_{-n+1} \\ +(r+s+t+u-1)W_{-n} + K_4 \end{array}}{r+s+t+u+v-1}$$

where

$$K_4 = W_4 + (1-r)W_3 + (1-r-s)W_2 + (1-r-s-t)W_1 + (1-r-s-t-u)W_0$$

(b) *If $(r - s + t - u + v + 1)(r + s + t + u + v - 1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\begin{array}{c} -(r+t+v)W_{-2n+3} + (r^2+s+u+rt+rv-1)W_{-2n+2} \\ -(t+v+ru-st-sv)W_{-2n+1} + (r^2-s^2+t^2+2s+u+2rt+rv-su+tv-1)W_{-2n} \\ +v(s+u-1)W_{-2n-1} + K_5 \end{array}}{(r-s+t-u+v+1)(r+s+t+u+v-1)}$$

where

$$\begin{aligned} K_5 &= (1-s-u)W_4 + (t+v+rs+ru)W_3 + (1-2s-u-r^2+s^2-rt-rv+su)W_2 \\ &\quad +(v+ru-sv+tu)W_1 + (1-2s-2u-r^2+s^2-t^2+u^2-2rt-rv+2su-tv)W_0 \end{aligned}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\begin{array}{c} (s+u-1)W_{-2n+3} - (t+v+rs+ru)W_{-2n+2} \\ +(2s+u+rt+rv-su+r^2-s^2-1)W_{-2n+1} - (v+ru-sv+tu)W_{-2n} \\ -v(r+t+v)W_{-2n-1} + K_6 \end{array}}{(r+s+t+u+v-1)(r-s+t-u+v+1)}$$

where

$$\begin{aligned} K_6 &= (r+t+v)W_4 + +(1-s\ -u\ -rt\ -rv\ -r^2)W_3 + (t\ +v\ +ru\ -st\ -sv)W_2 \\ &\quad +(1-u\ -2s-2rt\ -rv\ +su\ -tv\ -r^2\ +s^2\ -t^2\ )W_1 + (v\ -sv\ -uv)W_0 \end{aligned}$$

(c) *If* $s+u-1 \neq 0 \wedge r+t+v=0$ *then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\begin{array}{c} -W_{-2n+2} - (t\ +v)W_{-2n+1} + (s\ -1)W_{-2n} - vW_{-2n-1} + W_4 + (v+t)W_3 \\ +(1-s)W_2 + vW_1 + (1-s\ -u)W_0 \end{array}}{s+u-1}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{-W_{-2n+3} - (t+v)W_{-2n+2} + (s\ -1)W_{-2n+1} - vW_{-2n} + W_3 + (t\ +v)W_2 + (1-s)W_1 + vW_0}{s+u-1}.$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+5} = rW_{-n+4} + sW_{-n+3} + tW_{-n+2} + uW_{-n+1} + vW_{-n}$$

i.e.

$$vW_{-n} = W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - uW_{-n+1}$$

we obtain

$$\begin{aligned} vW_{-n} &= W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - uW_{-n+1} \\ vW_{-n+1} &= W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} \\ vW_{-n+2} &= W_{-n+7} - rW_{-n+6} - sW_{-n+5} - tW_{-n+4} - uW_{-n+3} \\ &\vdots \\ vW_{-6} &= W_{-1} - rW_{-2} - sW_{-3} - tW_{-4} - uW_{-5} \\ vW_{-5} &= W_0 - rW_{-1} - sW_{-2} - tW_{-3} - uW_{-4} \\ vW_{-4} &= W_1 - rW_0 - sW_{-1} - tW_{-2} - uW_{-3} \\ vW_{-3} &= W_2 - rW_1 - sW_0 - tW_{-1} - uW_{-2} \\ vW_{-2} &= W_3 - rW_2 - sW_1 - tW_0 - uW_{-1} \\ vW_{-1} &= W_4 - rW_3 - sW_2 - tW_1 - uW_0. \end{aligned}$$

If we add the above equations by side by, we obtain

$$\begin{aligned} v(\sum_{k=1}^{n} W_{-k}) &= (-W_{-n+4} - W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_4 + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) \\ &\quad -r(-W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) \\ &\quad -s(-W_{-n+2} - W_{-n+1} - W_{-n} + W_2 + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) \\ &\quad -t(-W_{-n+1} - W_{-n} + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) - u(-W_{-n} + W_0 + \sum_{k=1}^{n} W_{-k}) \end{aligned}$$

and then ve get (a).

(b) and (c) Using the recurrence relation

$$\begin{aligned} W_{-n+5} &= rW_{-n+4} + sW_{-n+3} + tW_{-n+2} + uW_{-n+1} + vW_{-n} \\ &\Rightarrow W_{-n} = \frac{1}{v}W_{-n+5} - \frac{u}{v}W_{-n+1} - \frac{t}{v}W_{-n+2} - \frac{s}{v}W_{-n+3} - \frac{r}{v}W_{-n+4} \end{aligned}$$

i.e.

$$uW_{-n+1} = W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - vW_{-n}$$

we obtain

$$\begin{aligned} uW_{-2n+1} &= W_{-2n+5} - rW_{-2n+4} - sW_{-2n+3} - tW_{-2n+2} - vW_{-2n} \\ uW_{-2n+3} &= W_{-2n+7} - rW_{-2n+6} - sW_{-2n+5} - tW_{-2n+4} - vW_{-2n+2} \\ uW_{-2n+5} &= W_{-2n+9} - rW_{-2n+8} - sW_{-2n+7} - tW_{-2n+6} - vW_{-2n+4} \\ uW_{-2n+7} &= W_{-2n+11} - rW_{-2n+10} - sW_{-2n+9} - tW_{-2n+8} - vW_{-2n+6} \\ &\vdots \\ uW_{-5} &= W_{-1} - rW_{-2} - sW_{-3} - tW_{-4} - vW_{-6} \\ uW_{-3} &= W_{1} - rW_{0} - sW_{-1} - tW_{-2} - vW_{-4} \\ uW_{-1} &= W_{3} - rW_{2} - sW_{1} - tW_{0} - vW_{-2}. \end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned} u\sum_{k=1}^{n} W_{-2k+1} &= (-W_{-2n+3} - W_{-2n+1} + W_3 + W_1 + \sum_{k=1}^{n} W_{-2k+1}) \\ &\quad -r(-W_{-2n+2} - W_{-2n} + W_0 + W_2 + \sum_{k=1}^{n} W_{-2k}) \\ &\quad -s(-W_{-2n+1} + W_1 + \sum_{k=1}^{n} W_{-2k+1}) \\ &\quad -t(-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - v(\sum_{k=1}^{n} W_{-2k}). \end{aligned} \tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+5} = rW_{-n+4} + sW_{-n+3} + tW_{-n+2} + uW_{-n+1} + vW_{-n}$$

i.e.

$$uW_{-n+1} = W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} - vW_{-n}$$

we obtain

$$\begin{aligned} uW_{-2n} &= W_{-2n+4} - rW_{-2n+3} - sW_{-2n+2} - tW_{-2n+1} - vW_{-2n-1} \\ uW_{-2n+2} &= W_{-2n+6} - rW_{-2n+5} - sW_{-2n+4} - tW_{-2n+3} - vW_{-2n+1} \\ uW_{-2n+4} &= W_{-2n+8} - rW_{-2n+7} - sW_{-2n+6} - tW_{-2n+5} - vW_{-2n+3} \\ uW_{-2n+6} &= W_{-2n+10} - rW_{-2n+9} - sW_{-2n+8} - tW_{-2n+7} - vW_{-2n+5} \\ &\vdots \\ uW_{-8} &= W_{-4} - rW_{-5} - sW_{-6} - tW_{-7} - vW_{-9} \\ uW_{-6} &= W_{-2} - rW_{-3} - sW_{-4} - tW_{-5} - vW_{-7} \\ uW_{-4} &= W_{0} - rW_{-1} - sW_{-2} - tW_{-3} - vW_{-5} \\ uW_{-2} &= W_{2} - rW_{1} - sW_{0} - tW_{-1} - vW_{-3}. \end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned} u\sum_{k=1}^{n}W_{-2k} &= (-W_{-2n+2}-W_{-2n}+W_{2}+W_{0}+\sum_{k=1}^{n}W_{-2k}) \\ &\quad -r(-W_{-2n+1}+W_{1}+\sum_{k=1}^{n}W_{-2k+1})-s(-W_{-2n}+W_{0}+\sum_{k=1}^{n}W_{-2k}) \\ &\quad -t(\sum_{k=1}^{n}W_{-2k+1})-v(W_{-2n-1}-W_{-1}+\sum_{k=1}^{n}W_{-2k+1}). \end{aligned}$$

Since

$$-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}+\frac{1}{v}W_{4}$$

it follows that

$$\begin{aligned} u\sum_{k=1}^{n}W_{-2k} &= (-W_{-2n+2}-W_{-2n}+W_{2}+W_{0}+\sum_{k=1}^{n}W_{-2k})-r(-W_{-2n+1}+W_{1}+\sum_{k=1}^{n}W_{-2k+1}) \quad (3.2) \\ &\quad -s(-W_{-2n}+W_{0}+\sum_{k=1}^{n}W_{-2k}) \\ &\quad -t(\sum_{k=1}^{n}W_{-2k+1})-v(W_{-2n-1}-(-\frac{u}{v}W_{0}-\frac{t}{v}W_{1}-\frac{s}{v}W_{2}-\frac{r}{v}W_{3}+\frac{1}{v}W_{4}) \\ &\quad +\sum_{k=1}^{n}W_{-2k+1}). \end{aligned}$$

Then, solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking $r=s=t=u=v=1$ in Theorem 3.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.1. *If $r=s=t=u=v=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n}W_{-k}=\frac{1}{4}(-W_{-n+4}+W_{-n+2}+2W_{-n+1}+3W_{-n}+W_{4}-W_{2}-2W_{1}-3W_{0})$.

(b) $\sum_{k=1}^{n}W_{-2k}=\frac{1}{8}(-3W_{-2n+3}+4W_{-2n+2}-W_{-2n+1}+6W_{-2n}+W_{-2n-1}-W_{4}+4W_{3}-3W_{2}+2W_{1}-5W_{0})$.

(c) $\sum_{k=1}^{n}W_{-2k+1}=\frac{1}{8}(W_{-2n+3}-4W_{-2n+2}+3W_{-2n+1}-2W_{-2n}-3W_{-2n-1}+3W_{4}-4W_{3}+W_{2}-6W_{1}-W_{0})$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Pentanacci numbers (take $W_n=P_n$ with $P_0=0,P_1=1,P_2=1,P_3=2,P_4=4$).

Corollary 3.2. *For $n\geq 1$, Pentanacci numbers have the following properties.*

(a) $\sum_{k=1}^{n}P_{-k}=\frac{1}{4}(-P_{-n+4}+P_{-n+2}+2P_{-n+1}+3P_{-n}+1)$.

(b) $\sum_{k=1}^{n}P_{-2k}=\frac{1}{8}(-3P_{-2n+3}+4P_{-2n+2}-P_{-2n+1}+6P_{-2n}+P_{-2n-1}+3)$.

(c) $\sum_{k=1}^{n}P_{-2k+1}=\frac{1}{8}(P_{-2n+3}-4P_{-2n+2}+3P_{-2n+1}-2P_{-2n}-3P_{-2n-1}-1)$

Taking $W_n=Q_n$ with $Q_0=5,Q_1=1,Q_2=3,Q_3=7,Q_4=15$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Pentanacci-Lucas numbers.

Corollary 3.3. *For $n\geq 1$, Pentanacci-Lucas numbers have the following properties.*

(a) $\sum_{k=1}^{n}Q_{-k}=\frac{1}{4}(-Q_{-n+4}+Q_{-n+2}+2Q_{-n+1}+3Q_{-n}-5)$.

(b) $\sum_{k=1}^{n}Q_{-2k}=\frac{1}{8}(-3Q_{-2n+3}+4Q_{-2n+2}-Q_{-2n+1}+6Q_{-2n}+Q_{-2n-1}-19)$.

(c) $\sum_{k=1}^{n}Q_{-2k+1}=\frac{1}{8}(Q_{-2n+3}-4Q_{-2n+2}+3Q_{-2n+1}-2Q_{-2n}-3Q_{-2n-1}+9)$

Taking $r=2, s=t=u=v=1$ in Theorem 3.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.2. *If $r=2, s=t=u=v=1$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}=\frac{1}{5}(-W_{-n+4}+W_{-n+3}+2W_{-n+2}+3W_{-n+1}+4W_{-n}+W_4-W_3-2W_2-3W_1-4W_0)$.

(b) $\sum_{k=1}^{n} W_{-2k}=\frac{1}{15}(-4W_{-2n+3}+9W_{-2n+2}-2W_{-2n+1}+12W_{-2n}+W_{-2n-1}-W_4+6W_3-8W_2+3W_1-11W_0)$.

(c) $\sum_{k=1}^{n} W_{-2k+1}=\frac{1}{15}(W_{-2n+3}-6W_{-2n+2}+8W_{-2n+1}-3W_{-2n}-4W_{-2n-1}+4W_4-9W_3+2W_2-12W_1-W_0)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth-order Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1, P_2=2, P_3=5, P_4=13$).

Corollary 3.4. *For $n\geq 1$, fifth-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} P_{-k}=\frac{1}{5}(-P_{-n+4}+P_{-n+3}+2P_{-n+2}+3P_{-n+1}+4P_{-n}+1)$.

(b) $\sum_{k=1}^{n} P_{-2k}=\frac{1}{15}(-4P_{-2n+3}+9P_{-2n+2}-2P_{-2n+1}+12P_{-2n}+P_{-2n-1}+4)$.

(c) $\sum_{k=1}^{n} P_{-2k+1}=\frac{1}{15}(P_{-2n+3}-6P_{-2n+2}+8P_{-2n+1}-3P_{-2n}-4P_{-2n-1}-1)$.

Taking $W_n=Q_n$ with $Q_0=5, Q_1=2, Q_2=6, Q_3=17, Q_4=46$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Pell-Lucas numbers.

Corollary 3.5. *For $n\geq 1$, fifth-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} Q_{-k}=\frac{1}{5}(-Q_{-n+4}+Q_{-n+3}+2Q_{-n+2}+3Q_{-n+1}+4Q_{-n}-9)$.

(b) $\sum_{k=1}^{n} Q_{-2k}=\frac{1}{15}(-4Q_{-2n+3}+9Q_{-2n+2}-2Q_{-2n+1}+12Q_{-2n}+Q_{-2n-1}-41)$.

(c) $\sum_{k=1}^{n} Q_{-2k+1}=\frac{1}{15}(Q_{-2n+3}-6Q_{-2n+2}+8Q_{-2n+1}-3Q_{-2n}-4Q_{-2n-1}+14)$.

Taking $r=s=t=1, u=1, v=2$ in Theorem 3.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 3.3. *If $r=s=t=1, u=1, v=2$ then for $n\geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} W_{-k}=\frac{1}{5}(-W_{-n+4}+W_{-n+2}+2W_{-n+1}+3W_{-n}+W_4-W_2-2W_1-3W_0)$

(b) $\sum_{k=1}^{n} W_{-2k}=\frac{1}{15}(-4W_{-2n+3}+5W_{-2n+2}-W_{-2n+1}+8W_{-2n}+2W_{-2n-1}-W_4+5W_3-4W_2+2W_1-7W_0)$

(c) $\sum_{k=1}^{n} W_{-2k+1}=\frac{1}{15}(W_{-2n+3}-5W_{-2n+2}+4W_{-2n+1}-2W_{-2n}-8W_{-2n-1}+4W_4-5W_3+W_2-8W_1-2W_0)$

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1, J_3=1, J_4=1$ in the last Proposition, we have the following Corollary which presents linear sum formulas of fifth-order Jacobsthal numbers.

Corollary 3.6. *For $n\geq 1$, fifth order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} J_{-k}=\frac{1}{5}(-J_{-n+4}+J_{-n+2}+2J_{-n+1}+3J_{-n}-2)$.

(b) $\sum_{k=1}^{n} J_{-2k}=\frac{1}{15}(-4J_{-2n+3}+5J_{-2n+2}-J_{-2n+1}+8J_{-2n}+2J_{-2n-1}+2)$.

(c) $\sum_{k=1}^{n} J_{-2k+1}=\frac{1}{15}(J_{-2n+3}-5J_{-2n+2}+4J_{-2n+1}-2J_{-2n}-8J_{-2n-1}-8)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of fifth order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5, j_3=10, j_4=20$).

Corollary 3.7. *For $n\geq 1$, fifth order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} j_{-k}=\frac{1}{5}(-j_{-n+4}+j_{-n+2}+2j_{-n+1}+3j_{-n}+7)$.

(b) $\sum_{k=1}^{n} j_{-2k}=\frac{1}{15}(-4j_{-2n+3}+5j_{-2n+2}-j_{-2n+1}+8j_{-2n}+2j_{-2n-1}-2)$.

(c) $\sum_{k=1}^{n} j_{-2k+1}=\frac{1}{15}(j_{-2n+3}-5j_{-2n+2}+4j_{-2n+1}-2j_{-2n}-8j_{-2n-1}+23)$.

4 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, linear sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written linear sum identities in terms of the generalized Pentanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Pentanacci, Pentanacci-Lucas, fifth order Pell, fifth order Pell-Lucas, fifth order Jacobsthal and fifth order Jacobsthal-Lucas sequences. All the listed identities in the corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can summarize the sections as follows:

- In section 1, we present some background about generalized Pentanacci numbers.
- In section 2, linear summation formulas have been presented for generalized Pentanacci numbers with positive subscripts. As special cases, linear summation formulas of Pentanacci, Pentanacci-Lucas, fifth order Pell, fifth order Pell-Lucas, fifth order Jacobsthal and fifth order Jacobsthal-Lucas numbers with positive subscripts have been given.
- In section 3, linear summation formulas have been presented for generalized Pentanacci numbers with negative subscripts. As special cases, linear summation formulas of Pentanacci, Pentanacci-Lucas, fifth order Pell, fifth order Pell-Lucas, fifth order Jacobsthal and fifth order Jacobsthal-Lucas numbers with negative subscripts have been given.

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Competing Interests

Author has declared that no competing interests exist.

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