Open Journal of Mathematical Sciences

PSR Press

Article

Book graphs are cycle antimagic

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Received: 12 March 2019; Accepted: 1 Jun 2019; Published: 30 Jun 2019.

Abstract: Let $G = (V, E)$ be a finite simple graph with $v = |V(G)|$ vertices and $e = |E(G)|$ edges. Further suppose that $\mathbb{H} := \{H_1, H_2, \ldots, H_t\}$ is a family of subgraphs of G. In case, each edge of $E(G)$ belongs to at least one of the subgraphs H_i from the family $\mathbb{H}$, we say G admits an edge-covering. When every subgraph H_i in $\mathbb{H}$ is isomorphic to a given graph H, then the graph G admits an *H-covering*. A graph G admitting H covering is called an (a, d)-H-antimagic if there is a bijection $\eta : V \cup E \to \{1, 2, \ldots, v + e\}$ such that for each subgraph H' of G isomorphic to H, the sum of labels of all the edges and vertices belongs to H' constitutes an arithmetic progression with the initial term a and the common difference d. For $\eta(V) = \{1, 2, 3, \ldots, v\}$, the graph G is said to be *super (a, d)-H-antimagic* and for $d = 0$ it is called *H-supermagic*. When the given graph H is a cycle C_m then H-covering is called C_m-covering and super (a, d)-H-antimagic labeling becomes super (a, d)-C_m-antimagic labeling. In this paper, we investigate the existence of super (a, d)-C_m-antimagic labeling of book graphs B_n, for $m = 4$, $n \geq 2$ and for differences $d = 1, 2, 3, \ldots, 13$.

Keywords: Book graph B_n, super (a, d)-C_4-antimagic.

MSC: 05C78, 05C70.

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1. Introduction

Let G be a finite and simple graph. A family of subgraphs $H_1, H_2, \ldots, H_t$ is defined as an *edge-covering* of G such that each edge of $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \ldots, t$. Then G admits an $(H_1, H_2, \ldots, H_t)$-*(edge) covering*. If every subgraph H_i is isomorphic to a given graph H, then the graph G admits an *H-covering*. A graph G admitting an H-covering is called *(a, d)-H-antimagic* if there exists a total labeling $\eta : V(G) \cup E(G) \to \{1, 2, \ldots, v + e\}$ such that for each subgraph H' of G isomorphic to H, the H'-weights,

$$wt_\eta(H') = \sum_{v \in V(H')} \eta(v) + \sum_{e \in E(H')} \eta(e),$$

constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers and t is the number of all subgraphs of G isomorphic to H. Moreover, G is said to be *super (a, d)-H-antimagic*, if the smallest possible labels appear on the vertices. If G is a super (a, d)-H-antimagic graph then the corresponding total labeling η is called the *super (a, d)-H-antimagic labeling*. For $d = 0$, the super (a, d)-H-antimagic graph is called *H-supermagic*.

The H-supermagic graph was first introduced by Gutiérrez *et al.* in [1]. They proved that the star $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$-supermagic for some h. They also proved that the path P_n and the cycle C_n are P_h-supermagic for some h. Lladó *et al.* [2] investigated C_n-supermagic graphs and proved that wheels, windmills, books and prisms are C_h-magic for some h. Some results on C_n-supermagic labelings of several classes of graphs can be found in [3]. Maryati *et al.* [4] gave P_h-supermagic labelings of shrubs, subdivision of shrubs and banana tree graphs. Other examples of H-supermagic graphs with different choices of H have been given by Jeyanthi *et al.* in [5]. Maryati *et al.* [6] investigated the G-supermagicness of a disjoint union of c copies of a graph G and showed that disjoint union of any paths is cP_h-supermagic for some c and h.

The (a, d)-H-antimagic labeling was introduced by Inayah *et al.* [7]. In [8] Inayah *et al.* investigated the super (a, d)-H-antimagic labelings for some shackles of a connected graph H.

For $H \cong K_2$, super (a,d)-H-antimagic labelings are also called super (a,d)-edge-antimagic total labelings. For further information on super edge-magic labelings, one can see [9–12].

The super (a,d)-H-antimagic labeling is related to a super d-antimagic labeling of type $(1,1,0)$ of a plane graph which is the generalization of a face-magic labeling introduced by Lih [13]. Further information on super d-antimagic labelings can be found in [14–16].

In [17], Awais *et al.* proved the existence of (a,d)-C_4-antimagic labeling of book graphs B_n (for difference $d = 0,1$) and of its disjoint union. In this paper, we study the existence of super (a,d)-C_4-antimagic labeling of book graphs B_n for differences $d = 1,2,3,\ldots,13$ and $n \geq 2$.

2. Super Cycle Antimagic Labeling

In this section, we discussed super (a,d)-C_4-antimagicness of *book graphs* for difference $d = 1,2,3,\ldots,13$.

Let $K_{1,n}$, $n \geq 2$ be a complete bipartite graph on $n+1$ vertices. The *book graph* B_n is a cartesian product of $K_{1,n}$ with K_2. i.e., $B_n \cong K_{1,n} \square K_2$. Clearly book graph B_n admits C_4-covering. The book graph B_n has the vertex set and edge set as

$$V(B_n) = \{y_1, y_2\} \cup \cup_{i=1}^{n} \{x_{(1,i)}, x_{(2,i)}\}$$

$$E(B_n) = \cup_{i=1}^{n} \{y_1 x_{(1,i)}, y_2 x_{(2,i)}, x_{(1,i)} x_{(2,i)}\} \cup \{y_1 y_2\}$$

respectively. It can be noted that $|V(B_n)| = 2(n+1)$ and $|E(B_n)| = 3n+1$.
Every $C_4^{(j)}, 1 \leq j \leq n$ in B_n has the vertex set: $V(C_4^{(j)}) = \{y_1, y_2, x_{(1,j)}, x_{(2,j)}\}$ and the edge set: $E(C_4^{(j)}) = \{y_1 y_2, y_1 x_{(1,j)}, y_2 x_{(2,j)}, x_{(1,j)} x_{(2,j)}\}$.
Under a total labeling ξ, the $C_4^{(j)}$-weights, $j = 1,\ldots,n$, would be:

$$\begin{aligned} wt_{\xi}(C_4^{(j)}) &= \sum_{v \in V(C_4^{(j)})} \xi(v) + \sum_{e \in E(C_4^{(j)})} \xi(e). \\ &= \sum_{k=1}^{2} \Big(\xi(y_k) + \xi(x_{(k,j)}) + \xi(y_k x_{(k,j)}) \Big) + \xi(y_1 y_2) + \xi(x_{(1,j)} x_{(2,j)}) \end{aligned} \tag{1}$$

Theorem 1. *For any integer $n \geq 2$, the book graph B_n admits super (a,d)-C_4-antimagic labeling for differences $d = 1,3,\ldots,13$.*

Proof. Under a labeling ξ, the set $\{y_1, y_2, y_1 y_2\}$, would be labeled as:

$$\xi(y_k) = k, \quad k = 1,2$$
$$\xi(y_1 y_2) = 2(n+1)+1$$

and therefore the partial sum of $wt_{\xi}(C_4^{(j)})$ would be

$$\xi(y_1) + \xi(y_2) + \xi(y_1 y_2) = 2(n+3). \tag{2}$$

For $d = 1,3,\ldots,9,13$

$$\xi_d(x_{(k,j)}) = \begin{cases} 2j+1, & k = 1 \\ 2(j+1), & k = 2 \end{cases}$$

$$\xi_{11}(x_{(k,j)}) = \begin{cases} 2+j, & k = 1 \\ n+2+j, & k = 2 \end{cases}$$

$$\xi_d(x_{(1,j)} x_{(2,j)}) = \begin{cases} 3n+4-j, & d = 1 \\ 2n+3+j, & d = 3,5,7,9 \\ 2n+1+3j, & d = 11,13 \end{cases}$$

$$\xi_d(y_k x_{(k,j)}) = \begin{cases} (k+3)n+4-j, & k=1,2 \quad d=1,3 \\ 5n+4-j, & k=1 \quad d=5 \\ 3n+3+j, & k=2 \quad d=5 \\ (k+2)n+3+j, & k=1,2 \quad d=7 \\ 3n+2j+k+1, & k=1,2 \quad d=9 \\ 3(n+j)-k, & k=1,2 \quad d=11 \\ 3(n+j)+k-1, & k=1,2 \quad d=13 \end{cases}$$

where indices j are taken modulo n.

Clearly $\xi(V(B_n)) = \{1, 2, \dots, 2(n+1)\}$. Therefore ξ is a super labeling together with $\xi(E(B_n)) = \{2(n+1)+1, 2(n+1)+2, \dots, 5n+3\}$ which shows ξ is a total labeling.

Using (1) and (2), $wt_{\xi_d}(C_4^{(j)})$ are:

$$wt_{\xi_d}(C_4^{(j)}) = \begin{cases} 14n+21+j, & d=1 \\ 13n+20+3j, & d=3 \\ 12n+19+5j, & d=5 \\ 11n+18+7j, & d=7 \\ 10n+17+9j, & d=9 \\ 11n+8+11j, & d=11 \\ 10n+11+13j, & d=13 \end{cases}$$

Clearly $wt_{\xi_d}(C_4^{(j)})$ constitutes arithmetic progression and therefore book graphs are super (a, d)-C_4-antimagic for $d = 1, 3, \dots, 13$. This completes the proof.
□

Theorem 2. *For any integer $n \geq 2$, the book graph B_n admits super (a, d)-C_4-antimagic labeling for differences $d = 2, 4, \dots, 10$.*

Proof. Case $n \equiv 0 \pmod 2$

For $d = 2, 4, 6, 8$ the labeling ξ for the set $\{y_1, y_2, y_1y_2\}$, would be labeled as:

$$\xi_d(y_1) = 1$$
$$\xi_d(y_2) = \frac{n}{2} + 2$$

$$\xi_d(y_1y_2) = 2n+3$$

and therefore the partial sum of $wt_\xi(C_4^{(j)})$ would be

$$\xi_d(y_1) + \xi_d(y_2) + \xi_d(y_1y_2) = \frac{5n}{2} + 6 \tag{3}$$

The remaining set of elements has the labeling ξ as:

$$\xi_d(x_{(k,j)}) = \begin{cases} 1+j, & k=1, \ j=1,2,\dots,\frac{n}{2} \\ 2j-\frac{n}{2}+1, & k=1, \ j=\frac{n}{2}+1,\dots,n \\ \frac{n}{2}+2(1+j), & k=2, \ j=1,2,\dots,\frac{n}{2} \\ n+2+j, & k=2, \ j=\frac{n}{2}+1,\dots,n \end{cases}$$

$$\xi_d(x_{(1,j)}x_{(2,j)}) = \begin{cases} 2(n+1)+1+j, & d=2 \\ 5n+4-j, & d=4 \\ 4n+3+j, & d=6,8 \end{cases}$$

$$\xi_d(y_k x_{(k,j)}) = \begin{cases} n(k+3)+4-j, & d=2 \\ n(k+1)+3+j, & d=4,6 \\ 2(n+j)+k+1, & d=8 \end{cases}$$

For difference $d = 10$ the labeling ξ is defined as:

$$\xi_d(y_1) = 1$$
$$\xi_d(y_2) = \frac{3n}{2} + 2$$
$$\xi_d(y_1y_2) = 2n+3$$

and the partial sum of $wt_\xi(C_4^{(j)})$ would be:

$$\xi_d(y_1) + \xi_d(y_2) + \xi_d(y_1y_2) = \frac{7n}{2} + 6 \tag{4}$$

$$\xi_d(x_{(k,j)}) = \begin{cases} 2j+1, & k=1,\ j=1,2,\dots,\frac{n}{2} \\ 2j-n, & k=1,\ j=\frac{n}{2}+1,\dots,n \\ \frac{3n}{2}+2-j, & k=2,\ j=1,2,\dots,\frac{n}{2} \\ \frac{5n}{2}+3-j, & k=2,\ j=\frac{n}{2}+1,\dots,n \end{cases}$$

$$\xi_d(x_{(1,j)}x_{(2,j)}) = 2n+1+3j$$
$$\xi_d(y_k x_{(k,j)}) = 2n+(k+1)+3j, \quad k=1,2$$

Clearly $\xi(V(B_n)) = \{1,2,\dots,2(n+1)\}$. Therefore ξ is a super labeling and together with $\xi(E(B_n)) = \{2(n+1)+1, 2(n+1)+2,\dots,5n+3\}$ which shows ξ is a total labeling.

Using (1), (3) and (4), $wt_\xi(C_4^{(j)})$ are:

$$wt_{\xi_d}(C_4^{(j)}) = \begin{cases} 14n+20+2j, & d=2 \\ 13n+19+4j, & d=4 \\ 12n+18+6j, & d=6 \\ 11n+17+8j, & d=8 \\ 11n+15+10j, & d=10 \end{cases}$$

Therefore $wt_{\xi_d}(C_4^{(j)})$ constitutes arithmetic progression for differences $d = 2,4,\dots,10$ when $n \equiv 0 \pmod 2$.

Case $n \equiv 1 \pmod 2$

For the set $\{y_1, y_2, y_1y_2\}$, labeling ξ would be:

$$\xi_d(y_1) = 1$$
$$\xi_d(y_2) = n+2$$
$$\xi_d(y_1y_2) = 2n+3$$

and therefore the partial sum of $wt_\xi(C_4^{(j)})$ would be

$$\xi_d(y_1) + \xi_d(y_2) + \xi_d(y_1y_2) = 3(n+2) \tag{5}$$

For differences $d = 2, 4, 6, 10$

$$\xi_d(x_{(k,j)}) = \begin{cases} 2j, & k=1,\ j=1,2,\ldots,\frac{n+1}{2} \\ 2j-n, & k=1,\ j=\frac{n+1}{2}+1,\ldots,n \\ 3\left(\frac{n+1}{2}\right)+2-j, & k=2,\ j=1,2,\ldots,\frac{n+1}{2} \\ 5\left(\frac{n+1}{2}\right)+1-j, & k=2,\ j=\frac{n+1}{2}+1,\ldots,n \end{cases}$$

and for differences $d = 8$

$$\xi_d(x_{(k,j)}) = \begin{cases} n+2-2j, & k=1,\ j=1,2,\ldots,\frac{n-1}{2} \\ 2(n+1)-2j, & k=1,\ j=\frac{n+1}{2},\ldots,n \\ 3\left(\frac{n+1}{2}\right)+1+j, & k=2,\ j=1,2,\ldots,\frac{n-1}{2} \\ \frac{n+1}{2}+2+j, & k=2,\ j=\frac{n+1}{2},\ldots,n \end{cases}$$

For differences $d = 2, 4, \ldots, 10$, the set of edges has the labeling ξ defined as:

$$\xi_d(x_{(1,j)}x_{(2,j)}) = \begin{cases} 5n+4-j, & d=2 \\ 4n+3+j, & d=4,6 \\ 2n+3+3j, & d=8,10 \end{cases}$$

$$\xi_d(y_k x_{(k,j)}) = \begin{cases} n(k+1)+3+j, & k=1,2,\ d=2,4 \\ 2(n+j)+k+1, & k=1,2,\ d=6 \\ 2n+k+3j, & d=8,10 \end{cases}$$

Clearly $\xi(V(B_n)) = \{1, 2, \ldots, 2(n+1)\}$. Therefore ξ is a super labeling together with $\xi(E(B_n)) = \{2(n+1)+1, 2(n+1)+2, \ldots, 5n+3\}$ which shows ξ is a total labeling.

Using (1) and (5), $wt_{\xi}(C_4^{(j)})$ are:

$$wt_{\xi_d}(C_4^{(j)}) = \begin{cases} \frac{27n+33}{2}+2j, & d=2 \\ \frac{25n+31}{2}+4j, & d=4 \\ \frac{23n+29}{2}+6j, & d=6 \\ \frac{21n+27}{2}+8j, & d=8 \\ \frac{19n+25}{2}+4j, & d=10 \end{cases}$$

Therefore $wt_{\xi_d}(C_4^{(j)})$ constitute arithmetic progression for differences $d = 2, 4, \ldots, 10$ when $n \equiv 1 \pmod 2$. Hence book graphs are super (a, d)-C_4-antimagic for $d = 2, 4, \ldots, 10$. This completes the proof.
□

Theorem 3. *For any integer $n \geq 2$, the book graph B_n admits super $(a, 12)$-C_4-antimagic labeling.*

Proof. Case $n \equiv 0 \pmod 2$

Under a labeling ξ, the set $\{y_1, y_2, y_1y_2\}$, would be labeled as:

$$\xi_{12}(y_1) = 1$$
$$\xi_{12}(y_2) = \frac{n+4}{2}$$
$$\xi_{12}(y_1y_2) = 2n+3$$

and therefore the partial sum of $wt_{\xi}(C_4^{(j)})$ would be

$$\xi_{12}(y_1) + \xi_d(y_2) + \xi_d(y_1y_2) = \frac{5n+12}{2} \tag{6}$$

$$\xi_{12}(x_{(k,j)}) = \begin{cases} 1+j & k=1,\ j=1,2,\ldots,\frac{n}{2} \\ 2j+1-\frac{n}{2} & k=1,\ j=\frac{n}{2}+1,\ldots,n \\ \frac{n}{2}+2(1+j), & k=2,\ j=1,2,\ldots,\frac{n}{2} \\ n+2+j, & k=2,\ j=\frac{n}{2}+1,\ldots,n \end{cases}$$

$$\xi_{12}(y_k x_{(k,j)}) = 2(n+k)+3j-1 \qquad k=1,2$$
$$\xi_{12}(x_{(1,j)}x_{(2,j)}) = 2(n+1)+3j$$

where indices j are taken modulo n.

Case $n \equiv 1 \pmod 2$

Under a labeling ξ, the set $\{y_1, y_2, y_1y_2\}$, would be labeled as:

$$\xi_{12}(y_k) = \frac{3}{2}(n-1)+2k$$

$$\xi_{12}(y_1y_2) = 2n+3$$

and therefore the partial sum of $wt_{\xi}(C_4^{(j)})$ would be

$$\xi_{12}(y_1) + \xi_d(y_2) + \xi_d(y_1y_2) = 5n+6 \tag{7}$$

$$\xi_{12}(x_{(k,j)}) = \begin{cases} j & k=1,\ j=1,2,\ldots,\frac{n+1}{2} \\ 2j-\frac{n+3}{2} & k=1,\ j=\frac{n+1}{2}+1,\ldots,n \\ \frac{n+1}{2}+2j, & k=2,\ j=\frac{n}{2}+1,\ldots,n \\ n+2+j, & k=2,\ j=\frac{n}{2}+1,\ldots,n \end{cases}$$

$$\xi_{12}(y_k x_{(k,j)}) = 2n+k+3j \qquad k=1,2$$
$$\xi_{12}(x_{(1,j)}x_{(2,j)}) = 2n+3(1+j)$$

where indices j are taken modulo n.

Clearly $\xi(V(B_n)) = \{1,2,\ldots,2(n+1)\}$. Therefore ξ is a super labeling together with $\xi(E(B_n)) = \{2(n+1)+1, 2(n+1)+2, \ldots, 5n+3\}$ which shows ξ is a total labeling.

Using (1), (6) and (7), $wt_{\xi}(C_4^{(j)})$ are:

$$wt_{\xi_{12}}(C_4^{(j)}) = \begin{cases} 3(3n+5)+12j & n \equiv 0 \pmod 2 \\ \frac{23n+25}{2}+12j & n \equiv 1 \pmod 2 \end{cases}$$

Hence book graphs are super $(a,12)$-C_4-antimagic. This completes the proof.
□

Acknowledgments: The author wishes to express his profound gratitude to the reviewers for their useful comments on the manuscript.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] Gutiérrez, A., & Lladó, A. (2005). Magic coverings. *Journal of Combinatorial Mathematics and Combinatorial Computing, 55*(2005), 43-56.

[2] Lladó, A., & Moragas, J. (2007). Cycle-magic graphs. *Discrete Mathematics, 307*(23), 2925-2933.

[3] Ngurah, A. A. G., Salman, A. N. M., & Susilowati, L. (2010). H-supermagic labelings of graphs. Discrete Mathematics, 310(8), 1293-1300.

[4] Maryati, T. K., Baskoro, E. T., & Salman, A. N. M. (2008). P_h-(super)magic labelings of some trees. *Journal of Combinatorial Mathematics and Combinatorial Computing, 65*, 198-204.

[5] Jeyanthi, P., & Selvagopal, P. (2010). More classes of H-supermagic graphs. *International Jornal of Algorithms, Computing and Mathematics, 3*(1), 93-108.

[6] Maryati, T. K., Salman, A. N. M., & Baskoro, E. T. (2013). Supermagic coverings of the disjoint union of graphs and amalgamations. *Discrete Mathematics, 313*(4), 397-405.

[7] Inayah, N., Salman, A. N. M., & Simanjuntak, R. (2009). On (a,d)-H-antimagic coverings of graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing, 71*, 273-281.

[8] Inayah, N., Simanjuntak, R., Salman, A. N. M., & Syuhada, K. I. A. (2013). Super (a,d)-H-antimagic total labelings for shackles of a connected graph H. *Australasian Journal of Combinatorics, 57*, 127-138.

[9] Wallis, W. D. (2001). *Magic graphs*. Birkhäuser, New York.

[10] Bača, M., Lin, Y. Q., Muntaner-Batle, F. A., & Rius-Font, M. (2009). Strong labelings of linear forests. *Acta Mathematica Sinica, English Series, 25*(12), 1951-1964.

[11] Baca, M., & Miller, M. (2008). *Super edge-antimagic graphs: A wealth of problems and some solutions*. Universal-Publishers.

[12] Figueroa-Centeno, R. M., Ichishima, R., & Muntaner-Batle, F. A. (2001). The place of super edge-magic labelings among other classes of labelings. *Discrete Mathematics, 231*(1-3), 153-168.

[13] Lih, K. W. (1983). On magic and consecutive labelings of plane graphs. *Utilitas Math, 24*, 165-197.

[14] Bača, M., Miller, M., Phanalasy, O., & Semaničová-Feňovčíková, A. (2010). Super d-antimagic labelings of disconnected plane graphs. *Acta Mathematica Sinica, English Series, 26*(12), 2283-2294.

[15] Bača, M., Brankovic, L., Semaničová-Feňovčíková, A., Labelings of plane graphs containing Hamilton path, Acta Math. Sinica - English Series, 2011, 27(4), 701–714.

[16] Bača, M., Numan, M., & Shabbir, A. (2013). Labelings of type (1, 1, 1) for toroidal fullerenes. *Turkish Journal of Mathematics, 37*(6), 899-907.

[17] Umar, M. A., Javed, M. A., Hussain, M., & Ali, B. R. (2018).Super (a,d)-C_4-antimagicness of book graphs. *Open Journla of Mathematical Sciences, 2*, 115-121.

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